

VARIATIONAL REPRESENTATIONS OF VARADHAN FUNCTIONALS

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ABSTRACT. Motivated by the theory of large deviations, we introduce a class of non-negative non-linear functionals that have a variational “rate function” representation.

1. INTRODUCTION

Let (\mathbf{X}, d) be a Polish space with metric $d(\cdot)$ and let $\mathbf{C}_b(\mathbf{X})$ denote the space of all bounded continuous functions $F : \mathbf{X} \rightarrow \mathbb{R}$. In his work on large deviations of probability measures μ_n , Varadhan [12] introduced a class of non-linear functionals \mathbb{L} defined by

$$(1) \quad \mathbb{L}(F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_{\mathbf{X}} \exp(nF(\mathbf{x})) d\mu_n$$

and used the large deviations principle of μ_n to prove the variational representation

$$(2) \quad \mathbb{L}(F) = L_0 + \sup_{\mathbf{x} \in \mathbf{X}} \{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\},$$

where $\mathbb{I} : \mathbf{X} \rightarrow [0, \infty]$ is the *rate function* governing the large deviations, and $L_0 := \mathbb{L}(0) = 0$.

Several authors [1, 3, 4, 9, 10, 11] abstracted non-probabilistic components from the theory of large deviations. In particular, in [3], see also [10, Theorem 3.1] we give conditions which imply the rate function representation (2) when the limit (1) exists, and we show that the rate function is determined from the dual formula

$$(3) \quad \mathbb{I}(\mathbf{x}) = \mathbb{L}(0) + \sup_{F \in \mathbf{C}_b(\mathbf{X})} \{F(\mathbf{x}) - \mathbb{L}(F)\}.$$

In fact, one can reverse Varadhan’s approach, and show that large deviations of probability measures μ_n follow from the variational representation (2) for (1), see [8, Theorem 1.2.3]. In this context we have $\mu_n(\mathbf{X}) = 1$ which implies $\mathbb{L}(0) = 0$ in (3) and correspondingly $L_0 = 0$ in (2).

“Asymptotic values” in [3] are essentially what we call Varadhan Functionals here; the theorems in that paper are not entirely satisfying because the assumptions are in terms of the underlying probability measures. In this paper we present a more satisfying approach which relies on the theory of probability for motivation purposes only.

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Definition 1.1. A function $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \rightarrow \mathbb{R}$ is a Varadhan Functional if the following conditions are satisfied.

- (4) If $F \leq G$ then $\mathbb{L}(F) \leq \mathbb{L}(G)$ for all $F, G \in \mathbf{C}_b(\mathbf{X})$
 (5) $\mathbb{L}(F + \text{const}) = \mathbb{L}(F) + \text{const}$ for all $F \in \mathbf{C}_b(\mathbf{X})$, $\text{const} \in \mathbb{R}$

Expression (1) provides an example of Varadhan Functional, if the limit exists. Another example is given by variational representation (2).

Condition (4) is equivalent to $\mathbb{L}(F \vee G) \geq \mathbb{L}(F) \vee \mathbb{L}(G)$, where $a \vee b$ denotes the maximum of two numbers. Varadhan Functionals like (1) satisfy a stronger condition.

Definition 1.2. A Varadhan Functional \mathbb{L} is *maximal* if $\mathbb{L}(\cdot)$ is a lattice homomorphism

$$(6) \quad \mathbb{L}(F \vee G) = \mathbb{L}(F) \vee \mathbb{L}(G).$$

It is easy to see that each Varadhan Functional $\mathbb{L}(\cdot)$ satisfies the Lipschitz condition $|\mathbb{L}(F) - \mathbb{L}(G)| \leq \|F - G\|_\infty$, compare (9). Thus \mathbb{L} is a continuous mapping from the Banach space $\mathbf{C}_b(\mathbf{X})$ of all bounded continuous functions into the real line. We will need the following stronger continuity assumption, motivated by the definition of the countable additivity of measures.

Definition 1.3. A Varadhan Functional is σ -continuous if the following condition is satisfied.

$$(7) \quad \text{If } F_n \searrow 0 \text{ then } \mathbb{L}(F_n) \rightarrow \mathbb{L}(0).$$

Notice that if \mathbf{X} is compact, then by Dini's theorem and the Lipschitz property, all Varadhan Functionals are σ -continuous.

Maximal Varadhan Functionals are convex; this follows from the proof of Theorem 2.1, which shows that formula (2) holds true for all Varadhan Functionals when the supremum is extended to all \mathbf{x} in the Čech-Stone compactification of \mathbf{X} .

A simple example of convex and maximal but not σ -continuous Varadhan Functional is $\mathbb{L}(F) = \limsup_{x \rightarrow \infty} F(x)$, where $F \in \mathbf{C}_b(\mathbb{R})$. This Varadhan Functional cannot be represented by variational formula (2). Indeed, (2) implies that $\mathbb{L}(\mathbf{x}) \geq F(\mathbf{x}) - \mathbb{L}(F) = F(\mathbf{x})$ for all $F \in \mathbf{C}_b(\mathbb{R})$ that vanish at ∞ ; hence $\mathbb{L}(\mathbf{x}) = \infty$ for all $\mathbf{x} \in \mathbb{R}$ and (2) gives $\mathbb{L}(F) = -\infty$ for all $F \in \mathbf{C}_b(\mathbb{R})$, a contradiction.

An example of a convex and σ -continuous but not maximal Varadhan Functional is $\mathbb{L}(F) = \log \int_{\mathbf{X}} \exp F(\mathbf{x}) \nu(d\mathbf{x})$, where ν is a finite non-negative measure.

2. VARIATIONAL REPRESENTATIONS

The main result of this paper is the following.

Theorem 2.1. *If a maximal Varadhan Functional $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \rightarrow \mathbb{R}$ is σ -continuous, then there is $L_0 \in \mathbb{R}$ such that variational representation (2) holds true and the rate function $\mathbb{I} : \mathbf{X} \rightarrow [0, \infty]$ is given by the dual formula (3). Furthermore, $\mathbb{I}(\cdot)$ is a tight rate function: sets $\mathbb{I}^{-1}([0, a]) \subset \mathbf{X}$ are compact for all $a > 0$.*

The next result is closely related to Bryc [3, Theorem T.1.1] and Deuschel & Stroock [6, Theorem 5.1.6]. Denote by $\mathcal{P}(\mathbf{X})$ the metric space (with Prokhorov metric) of all probability measures on a Polish space \mathbf{X} with the Borel σ -field generated by all open sets.

Theorem 2.2. *If a convex Varadhan Functional $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \rightarrow \mathbb{R}$ is σ -continuous, then there is a lower semicontinuous function $\mathbb{J} : \mathcal{P}(\mathbf{X}) \rightarrow [0, \infty]$ and a constant L_0 such that*

$$(8) \quad \mathbb{L}(F) = L_0 + \sup_{\mu \in \mathcal{P}} \left\{ \int F d\mu - \mathbb{J}(\mu) \right\}$$

for all bounded continuous functions F .

A well known example in large deviations is the convex σ -continuous functional $\mathbb{L}(F) := \log \int \exp F(\mathbf{x}) \nu(d\mathbf{x})$ with the rate function in (8) given by the relative entropy functional

$$\mathbb{J}(\mu) = \begin{cases} \int \log \frac{d\mu}{d\nu} d\mu & \text{if } \mu \ll \nu \text{ is absolutely continuous} \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.1. Deuschel & Stroock [6, Section 5.1] consider convex functionals $\Phi : \mathbf{C}_b(\mathbf{X}) \rightarrow \mathbb{R}$ such that $\Phi(\text{const}) = \text{const}$. Such functionals satisfy condition (5). Indeed, write $F + \text{const}$ as a convex combination

$$F + \text{const} = (1 - \theta)F + \frac{\theta}{2} \left(\frac{2\text{const}}{\theta} \right) + \frac{\theta}{2}(2F),$$

where $0 < \theta < 1$. Using convexity and $\Phi(\text{const}) = \text{const}$ we get $\Phi(F + \text{const}) \leq \Phi(F) + \text{const} + \theta(\frac{\Phi(2F)}{2} - \Phi(F))$. Since $\theta > 0$ is arbitrary this proves that $\Phi(F + \text{const}) \leq \Phi(F) + \text{const}$. By routine symmetry considerations (replacing $F \mapsto F - \text{const}$, and then $\text{const} \mapsto -\text{const}$), (5) follows.

3. PROOFS

Let $L_0 := \mathbb{L}(0)$. Passing to $\mathbb{L}'(F) := \mathbb{L}(F) - L_0$ if necessary, without losing generality we assume $\mathbb{L}(0) = 0$.

Lemma 3.1. *Let $\hat{\mathbf{X}}$ be a compact Hausdorff space. Suppose $\mathbf{X} \subset \hat{\mathbf{X}}$ is a separable metric space in the relative topology. If $\mathbf{x}_0 \in \hat{\mathbf{X}} \setminus \mathbf{X}$ then there are bounded continuous functions $F_n : \hat{\mathbf{X}} \rightarrow \mathbb{R}$ such that*

- (i) $F_n(\mathbf{x}) \searrow 0$ for all $\mathbf{x} \in \mathbf{X}$.
- (ii) $F_n(\mathbf{x}_0) = 1$ for all $n \in \mathbb{N}$.

Proof. Since $\hat{\mathbf{X}}$ is Hausdorff, for every $\mathbf{x} \in \mathbf{X}$ there is an open set $U_{\mathbf{x}} \ni \mathbf{x}$ such that its closure $\bar{U}_{\mathbf{x}}$ does not contain \mathbf{x}_0 .

By Lindelöf property for separable metric space \mathbf{X} , there is a countable subcover $\{U_n\}$ of $\{U_{\mathbf{x}}\}$.

A compact Hausdorff space $\hat{\mathbf{X}}$ is normal. So there are continuous functions $\phi_n : \hat{\mathbf{X}} \rightarrow \mathbb{R}$ such that $\phi_n|_{\bar{U}_n} = 0$ and $\phi_n(\mathbf{x}_0) = 1$.

To end the proof take $F_n(\mathbf{x}) = \min_{1 \leq k \leq n} \phi_k(\mathbf{x})$.

□

The following lemma is contained implicitly in [3, Theorem T.1.2].

Lemma 3.2. *Theorem 2.1 holds true for compact \mathbf{X} .*

Proof. Let $\mathbb{I}(\cdot)$ be defined by (3). Thus $\mathbb{I}(\mathbf{x}) \geq F(\mathbf{x}) - \mathbb{L}(F)$ which implies $\mathbb{L}(F) \geq \sup_{\mathbf{x} \in \mathbf{X}} \{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\}$. To end the proof we need therefore to establish the converse inequality. Fix a bounded continuous function $F \in \mathbf{C}_b(\mathbf{X})$ and $\epsilon > 0$. Let $s =$

$\sup_{\mathbf{x} \in \mathbf{X}} \{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\}$. Clearly $F(\mathbf{x}) - \mathbb{I}(\mathbf{x}) \leq s \leq \mathbb{L}(F)$. By (3) again, for every $\mathbf{x} \in \mathbf{X}$, there is $F_{\mathbf{x}} \in \mathbf{C}_b(\mathbf{X})$ such that $\mathbb{I}(\mathbf{x}) < F_{\mathbf{x}}(\mathbf{x}) - \mathbb{L}(F_{\mathbf{x}}) + \epsilon$. Therefore

$$F(\mathbf{x}) \leq s + \mathbb{I}(\mathbf{x}) < s + \epsilon + F_{\mathbf{x}}(\mathbf{x}) - \mathbb{L}(F_{\mathbf{x}})$$

This means that the sets $U_{\mathbf{x}} = \{\mathbf{y} \in \mathbf{X} : F(\mathbf{y}) - F_{\mathbf{x}}(\mathbf{y}) < s + \epsilon - \mathbb{L}(F_{\mathbf{x}})\}$ form an open covering of \mathbf{X} . Using compactness of \mathbf{X} , we choose a finite covering $U_{\mathbf{x}(1)}, \dots, U_{\mathbf{x}(k)}$. Then, writing $F_i = F_{\mathbf{x}(i)}$ we have

$$F(\mathbf{x}) < \max_{1 \leq i \leq k} \{F_i(\mathbf{x}) - \mathbb{L}(F_i)\} + s + \epsilon$$

for all $\mathbf{x} \in \mathbf{X}$.

Using (4), (5), and (6) we have

$$\begin{aligned} \mathbb{L}(F) &\leq \mathbb{L} \left(\max_{1 \leq i \leq k} \{F_i - \mathbb{L}(F_i)\} + s + \epsilon \right) = \mathbb{L} \left(\max_i \{F_i - \mathbb{L}(F_i)\} \right) + s + \epsilon = \\ &\quad \max_i \{ \mathbb{L}(F_i - \mathbb{L}(F_i)) \} + s + \epsilon \end{aligned}$$

Since (5) implies $\mathbb{L}(F_i - \mathbb{L}(F_i)) = \mathbb{L}(F_i) - \mathbb{L}(F_i) = 0$ this shows that $s \leq \mathbb{L}(F) < s + \epsilon$. Therefore $\mathbb{L}(F) = s$, proving (2). \square

Proof of Theorem 2.1. Let $\hat{\mathbf{X}}$ be the Čech-Stone compactification of \mathbf{X} . Since the inclusion $\mathbf{X} \subset \hat{\mathbf{X}}$ is continuous, we define $\hat{\mathbb{L}} : \mathbf{C}_b(\hat{\mathbf{X}}) \rightarrow \mathbb{R}$ by $\hat{\mathbb{L}}(\hat{F}) := \mathbb{L}(\hat{F}|_{\mathbf{X}})$. It is clear that $\hat{\mathbb{L}}$ is a maximal Varadhan Functional, so by Lemma 3.2 there is $\mathbb{I} : \hat{\mathbf{X}} \rightarrow [0, \infty]$ such that $\hat{\mathbb{L}}(\hat{F}) = \sup\{\hat{F}(\mathbf{x}) - \mathbb{I}(\mathbf{x}) : \mathbf{x} \in \hat{\mathbf{X}}\}$.

Using σ -continuity(7) it is easy to check that $\mathbb{I}(\mathbf{x}) = \infty$ for all $\mathbf{x} \in \hat{\mathbf{X}} \setminus \mathbf{X}$. Indeed, given $\mathbf{x}_0 \in \hat{\mathbf{X}} \setminus \mathbf{X}$ by Lemma 3.1 there are $F_n \in \mathbf{C}_b(\hat{\mathbf{X}})$ such that $F_n \searrow 0$ on \mathbf{X} , but $F_n(\mathbf{x}_0) = C > 0$. Then from (3) we get $\mathbb{I}(\mathbf{x}_0) \geq \hat{\mathbb{L}}(0) + F_n(\mathbf{x}_0) - \hat{\mathbb{L}}(F_n) \rightarrow \hat{\mathbb{L}}(0) + C$. Since $C > 0$ is arbitrary, $\mathbb{I}(\mathbf{x}_0) = \infty$.

This shows that $\hat{\mathbb{L}}(\hat{F}) = \sup\{\hat{F}(\mathbf{x}) - \mathbb{I}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\}$ for all $\hat{F} \in \mathbf{C}_b(\hat{\mathbf{X}})$. It remains to observe that since $\hat{\mathbf{X}}$ is a Čech-Stone compactification, every function $F \in \mathbf{C}_b(\mathbf{X})$ is a restriction to \mathbf{X} of some $\hat{F} \in \mathbf{C}_b(\hat{\mathbf{X}})$, see [7, IV.6.22]. Therefore (2) holds true for all $F \in \mathbf{C}_b(\mathbf{X})$.

To prove that the rate function is tight, suppose that there is $a > 0$ such that $\mathbb{I}^{-1}[0, a]$ is not compact. Then there is $\delta > 0$ and a sequence $\mathbf{x}_n \in \mathbf{X}$ such that $d(\mathbf{x}_m, \mathbf{x}_n) > \delta$ for all $m \neq n$. Since Polish spaces have Lindelöf property, there is a countable number of open balls of radius $\delta/2$ which cover \mathbf{X} . For $k = 1, 2, \dots$, denote by $B_k \ni \mathbf{x}_k$ one of the balls that contain \mathbf{x}_k , and let ϕ_k be a bounded continuous function such that $\phi_k(\mathbf{x}_k) = 2a$ and $\phi_k = 0$ on the complement of B_k . Then $F_n = \max_{k \geq n} \phi_k \searrow 0$ pointwise. On the other hand (2) implies $\mathbb{L}(F_n) \geq L_0 + F_n(\mathbf{x}_n) - \mathbb{I}(\mathbf{x}_n) \geq L_0 + a$, contradicting (7). \square

Lemma 3.3. *If $\mathbb{L}(\cdot)$ is a Varadhan Functional then*

$$(9) \quad \inf_{\mathbf{x} \in \mathbf{X}} \{F(\mathbf{x}) - G(\mathbf{x})\} \leq \mathbb{L}(F) - \mathbb{L}(G)$$

Proof. Let $\text{const} = \inf_{\mathbf{x}} \{F(\mathbf{x}) - G(\mathbf{x})\}$. Clearly, $F \geq G + \text{const}$. By positivity condition (4) this implies $\mathbb{L}(F) \geq \mathbb{L}(G + \text{const}) = \mathbb{L}(G) + \text{const}$. \square

The next lemma is implicitly contained in the proof of [3, Theorem T.1.1]. Let $\mathcal{P}_a(\mathbf{X})$ denote all regular finitely-additive probability measures on \mathbf{X} with the Borel field.

Lemma 3.4. *If $\mathbb{L}(\cdot)$ is a convex Varadhan Functional on $\mathbf{C}_b(\mathbf{X})$, then there exist a lower semicontinuous function $\mathbb{J} : \mathcal{P}_a(\mathbf{X}) \rightarrow [0, \infty]$ such that*

$$(10) \quad \mathbb{L}(F) = \mathbb{L}(0) + \sup\{\mu(F) - \mathbb{J}(\mu) : \mu \in \mathcal{P}_a(\mathbf{X})\},$$

and the supremum is attained.

Proof. Let $\mathbb{J}(\cdot)$ be defined by

$$(11) \quad \mathbb{J}(\mu) = \mathbb{L}(0) + \sup\{\mu(F) - \mathbb{L}(F) : F \in \mathbf{C}_b(\mathbf{X})\}.$$

and fix $F_0 \in \mathbf{C}_b(\mathbf{X})$. Recall that throughout this proof we assume $\mathbb{L}(0) = 0$.

By the definition of $\mathbb{J}(\cdot)$, we need to show that

$$(12) \quad \mathbb{L}(F_0) = \sup_{\mu} \inf_F \{\mu(F_0) - \mu(F) + \mathbb{L}(F)\},$$

where the supremum is taken over all $\mu \in \mathcal{P}_a(\mathbf{X})$ and the infimum is taken over all $F \in \mathbf{C}_b(\mathbf{X})$. Moreover, since (11) implies that $\mathbb{J}(\mu) \geq \mu(F_0) - \mathbb{L}(F_0)$ for all $\mu \in \mathcal{P}_a(\mathbf{X})$, therefore $\mathbb{L}(F_0) \geq \sup_{\mu} \inf_F \{\mu(F_0) - \mu(F) + \mathbb{L}(F)\}$. Hence to prove (12), it remains to show that there is $\nu \in \mathcal{P}_a(\mathbf{X})$ such that

$$(13) \quad \mathbb{L}(F_0) \leq \nu(F_0) - \nu(F) + \mathbb{L}(F) \text{ for all } F \in \mathbf{C}_b(\mathbf{X}).$$

(also, for this ν , the supremum in (10) will be attained) To find ν , consider the following sets. Let

$$\mathcal{M} = \{F \in \mathbf{C}_b(\mathbf{X}) : \inf_{\mathbf{x}} [F(\mathbf{x}) - F_0(\mathbf{x})] > 0\}$$

and let \mathcal{N} be a set of all finite convex combinations of functions $g(\mathbf{x})$ of the form $g(\mathbf{x}) = F(\mathbf{x}) + \mathbb{L}(F_0) - \mathbb{L}(F)$, where $F \in \mathbf{C}_b(\mathbf{X})$.

It is easily seen from the definitions that \mathcal{M} and \mathcal{N} are convex; also $\mathcal{M} \subset \mathbf{C}_b(\mathbf{X})$ is non-empty since $1 + F_0 \in \mathcal{M}$, and open since $\{F : \inf_{\mathbf{x}} [F(\mathbf{x}) - F_0(\mathbf{x})] \leq 0\} \subset \mathbf{C}_b(\mathbf{X})$ is closed. Furthermore, \mathcal{M} and \mathcal{N} are disjoint. Indeed, take arbitrary

$$\mathcal{N} \ni g = \sum \alpha_k F_k + \mathbb{L}(F_0) - \sum \alpha_k \mathbb{L}(F_k).$$

Then

$$\begin{aligned} & \inf_x \{g(\mathbf{x}) - F_0(\mathbf{x})\} = \\ & \inf_x \left\{ \sum \alpha_k F_k(\mathbf{x}) - F_0(\mathbf{x}) \right\} - \sum \alpha_k \mathbb{L}(F_k) + \mathbb{L}(F_0) \leq \\ & \inf_x \left\{ \sum \alpha_k F_k(\mathbf{x}) - F_0(\mathbf{x}) \right\} - \mathbb{L}(\sum \alpha_k F_k) + \mathbb{L}(F_0) \leq 0, \end{aligned}$$

where the first inequality follows from the convexity of $\mathbb{L}(\cdot)$ and the second one follows from (9) applied to $F = \sum \alpha_k F_k(\mathbf{x})$ and $G = F_0$.

Therefore \mathcal{M} and \mathcal{N} can be separated, i. e. there is a non-zero linear functional $f^* \in \mathbf{C}_b^*(\mathbf{X})$ such that for some $\alpha \in \mathbb{R}$

$$(14) \quad f^*(\mathcal{N}) \leq \alpha < f^*(\mathcal{M}),$$

see e. g. [7, V. 2. 8]

Claim: f^* is non-negative.

Indeed, it is easily seen that $F_0(\cdot)$ belongs to \mathcal{N} , and, as a limit of $\epsilon + F_0(\mathbf{x})$ as $\epsilon \rightarrow 0$, F_0 is also in the closure of \mathcal{M} . Therefore by (14) we have $\alpha = f^*(F_0)$. To end the proof take arbitrary F with $\inf_{\mathbf{x}} F(\mathbf{x}) > 0$. Then $F + F_0 \in \mathcal{M}$ and by (14)

$$f^*(F) = f^*(F + F_0) - f^*(F_0) > \alpha - f^*(F_0) = 0.$$

This ends the proof of the claim.

Without loosing generality, we may assume $f^*(1) = 1$; then it is well known, see e. g. [2, Ch. 2 Section 4 Theorem 1], that $f^*(F) = \nu(F)$ for some $\nu \in \mathcal{P}_a(\mathbf{X})$; for regularity of ν consult [7, IV.6.2]. It remains to check that ν satisfies (13). To this end observe that since $F + \mathbb{L}(F_0) - \mathbb{L}(F) \in \mathcal{N}$, by (14) we have $\nu(F) + \mathbb{L}(F_0) - \mathbb{L}(F) \leq \alpha = \nu(F_0)$ for all $F \in \mathbf{C}_b(\mathbf{X})$. This ends the proof of (10). \square

Proof of Theorem 2.2. Lemma 3.4 gives the variational representation (10) with the supremum taken over a too large set. To end the proof we will show that $\mathbb{J}(\mu) = \infty$ on measures μ that fail to be countably-additive.

Suppose that μ is additive but not countably additive. Then Daniell-Stone theorem implies that there is $\delta > 0$ and a sequence $F_n \searrow 0$ of bounded continuous functions such that $\int F_n d\mu > \delta > 0$ for all n . By (11) and σ -continuity $\mathbb{J}(\mu) \geq \mathbb{L}(0) + C \int F_n d\mu - \mathbb{L}(CF_n) \geq \mathbb{L}(0) + C\delta - \mathbb{L}(CF_n) \rightarrow \mathbb{L}(0) + C\delta$. Since $C > 0$ is arbitrary, therefore $\mathbb{J}(\mu) = \infty$ for all μ that are additive but not countably-additive. Thus (10) implies (8). \square

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